The dollar game

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University of Michigan

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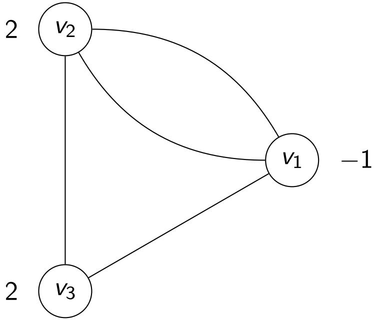
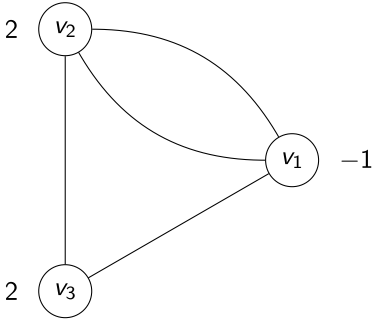
Section 1

The basics

The rules



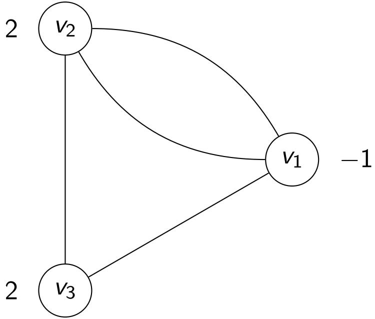
The rules



Each vertex is assigned an integer amount of wealth. We say that v_1 is *in debt*.

The rules

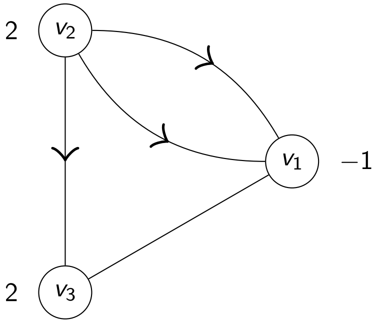
Lending and **borrowing** moves redistribute wealth.



The rules

Lending and **borrowing** moves redistribute wealth.

To lend at v_2:



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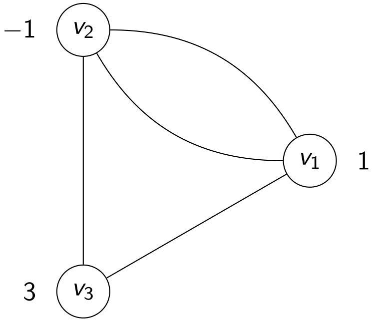
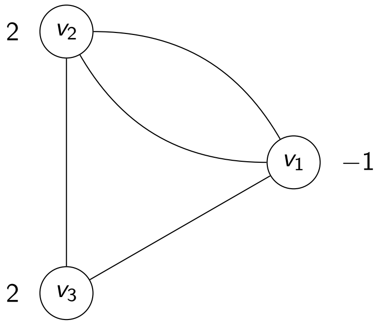


Borrowing moves defined analogously.

The rules

Question (The dollar game)

Does there exist a sequence of lending/borrowing moves that brings every vertex out of debt?



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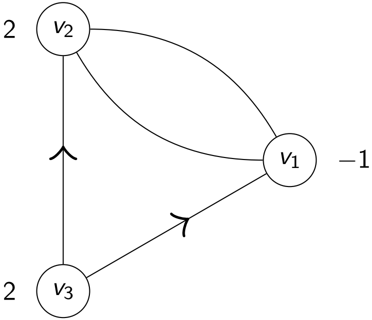

Yes! Lend at v_3.

The rules

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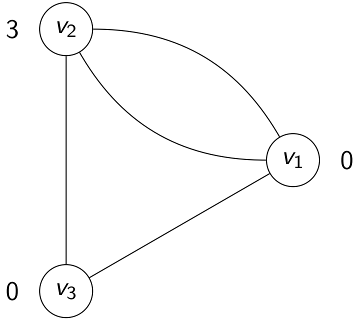


Yes! Lend at v_3.

A group-theoretic formulation

Definition

Take a graph G with vertices V and define $D : V \to \mathbb{Z}$. Then a **divisor** on G is an element of the abelian group:

$$\mathrm{Div}(G) = \mathbb{Z}V = \left\{ \sum_{v \in V} D(v)v \right\}.$$

A divisor assigns each vertex a coefficient representing its integer dollar wealth.

The divisor corresponding to $(v_1, v_2, v_3) \mapsto (-1, 2, 2)$ is $-v_1 + 2v_2 + 2v_3$.

A group-theoretic formulation

Definition

Take a graph G. Given $D \in \mathrm{Div}(G)$, a **lending move** at $v \in V$ produces $D' \in \mathrm{Div}(G)$ as follows:

$$D - \sum_{vw \in E} (v - w) =: D'.$$

A **borrowing move** at v produces

$$D - \sum_{vw \in E} (w - v) =: D'.$$

To lend at v_3 on $-v_1 + 2v_2 + 2v_3$, subtract $(2v_3 - v_1 - v_2)$ to obtain

$$-v_1 + 2v_2 + 2v_3 - (2v_3 - v_1 - v_2) = 3v_2.$$

Why this formulation?

- The *abelian property* is immediate.
- Allows us to think about the following:

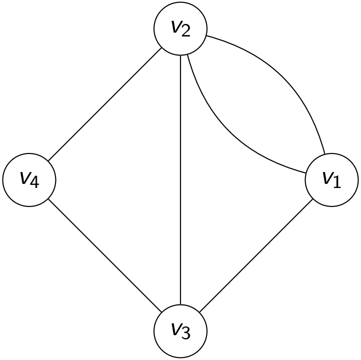
Proposition

For $D \in \mathrm{Div}(G)$, lending at $v' \in V$ is equivalent to borrowing at all $v \in V \backslash \{v'\}$.

We say that $V \backslash \{v'\}$ was *set-fired*.

Set-firing

What happens if we fire $\{v_2, v_3, v_4\}$?



Set-firing

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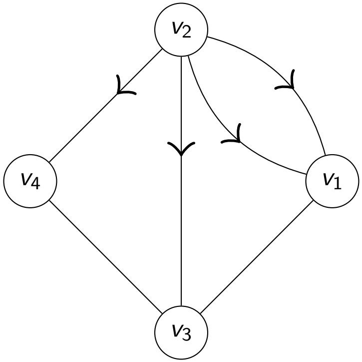

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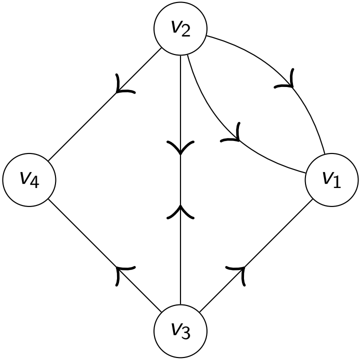

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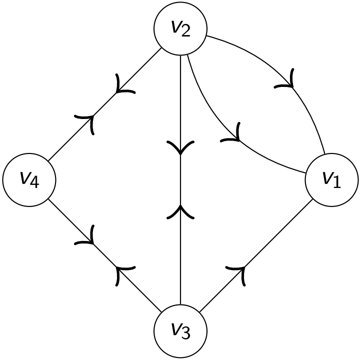

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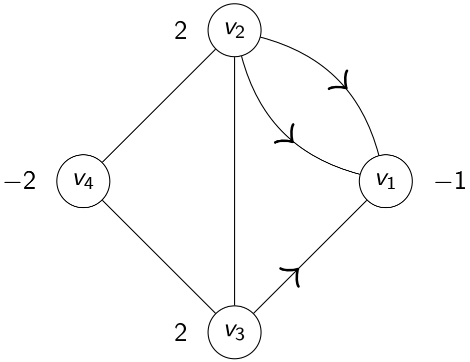


Only the "border vertices" feel a net effect.

A group-theoretic formulation

Definition

Take $D, D' \in \mathrm{Div}(G)$. Then D is **linearly equivalent** to D', or $D \sim D'$, if some sequence of lending moves on D produces D'.

Definition

The **divisor class** of $D \in \mathrm{Div}(G)$ is given by

$$[D] \coloneqq \{D' \in \mathrm{Div}(G) : D \sim D'\}.$$

Definition

An **effective** divisor D is one such that $D(v) \geq 0$ for all $v \in V$.

A group-theoretic formulation

Question (The dollar game)

Is a divisor D linearly equivalent to an effective divisor? If so, we say D is *winnable*; otherwise, it is *unwinnable*.

Section 2

Winning the dollar game

q-reduction

Take a divisor D. We perform *q-reduction* at the source vertex q as follows:

Algorithm (q-reduction)

1. Pick a vertex q and have the other vertices borrow from it until only q is in debt.
2. Find and lend from a set of vertices that will not leave behind debt. Call this a *legal* firing move.
3. Repeat step 2 for as long as possible.

q-reduction

Theorem

1. *The q-reduction algorithm always terminates.*
2. *The q-reduced divisor is unique to D and q.*
3. *D is winnable if and only if q is not in debt after q-reduction.*

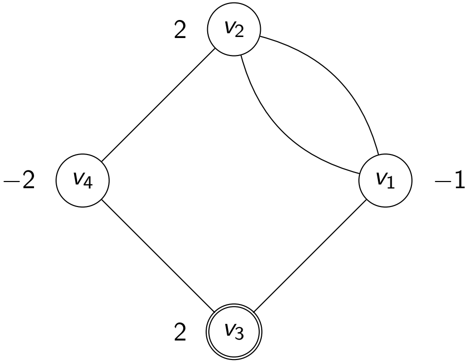
Proof of 3.

Backward direction: immediate.

Forward direction: let $D \sim E$ for effective E. Then q-reduce D and E to obtain D', E'. By uniqueness, $D' = E'$ since they are determined by the same D and q, so D' is effective. Therefore, D is winnable. □

q-reduction

Is this divisor winnable? q-reduce at v_3:



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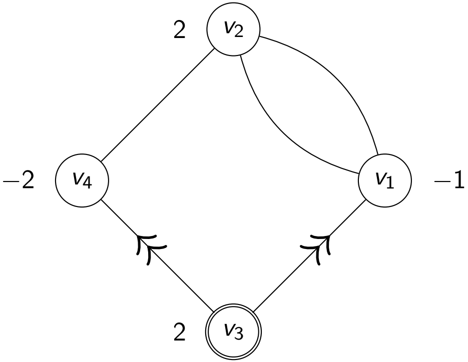

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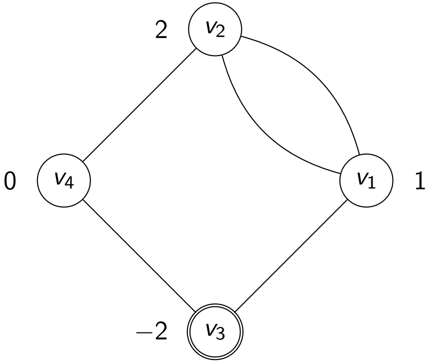

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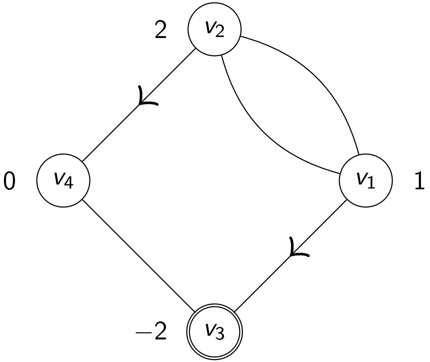

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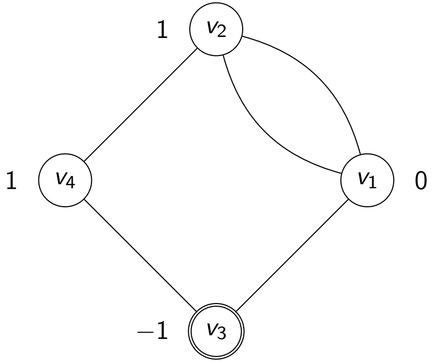

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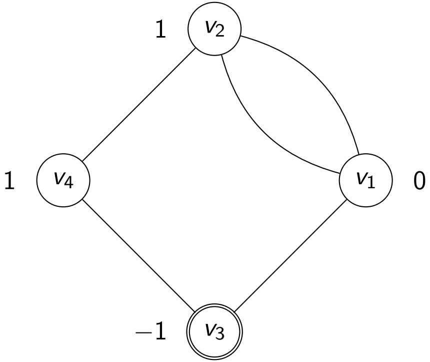


No set can be fired without leaving behind debt. Unwinnable!

Section 3

Connection to algebraic geometry

Divisors on varieties

Definition

An **algebraic variety** is the set of solutions to a system of polynomial equations over a field.

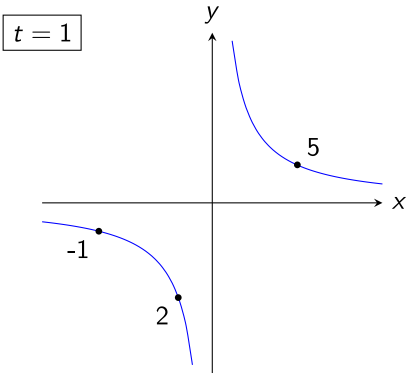
Example

- The variety corresponding to the circle in $\mathbb{R}^2$ is $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}$
- The variety $\{(x, y) \in \mathbb{R}^2 : xy - t = 0\}$ describes a hyperbola.

Divisors on varieties

Definition

A **divisor** on a curve is a member of the free abelian group on the points of the curve.



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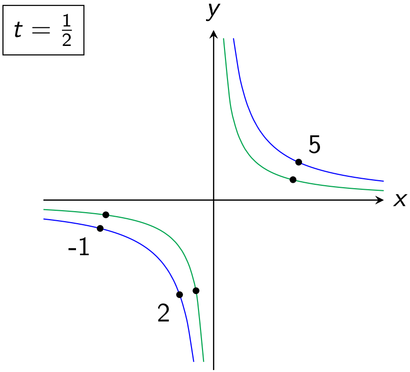

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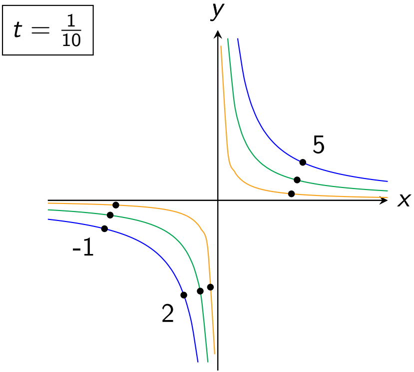

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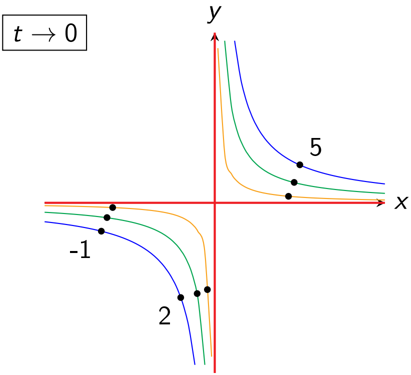
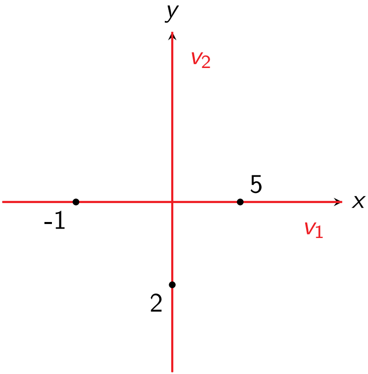


This is called the **special fiber**.

Divisors on varieties



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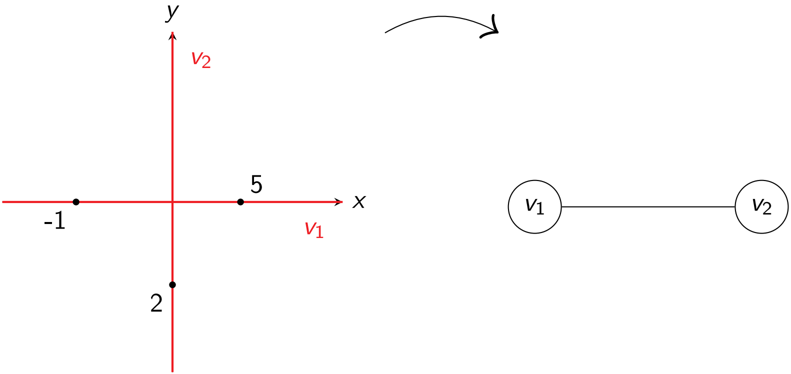

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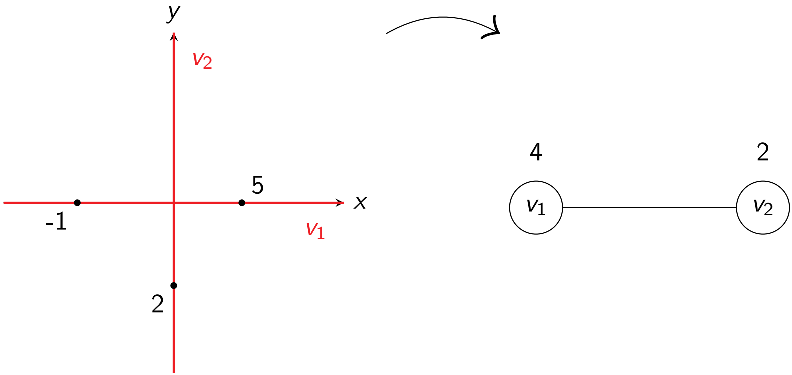

Proposition

Given a curve X and its dual graph G, there exists a surjective group homomorphism $\rho : \mathrm{Div}(X) \to \mathrm{Div}(G)$.

Thank you!