Representations of $\mathfrak{sl}(2;\mathbb{C})$

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Section 1

Lie groups and Lie algebras

Matrix Lie groups

Definition

The **general linear group** over the complex numbers is denoted $GL(n;\mathbb{C})$ and consists of all invertible $n \times n$ matrices with complex entries.

Definition

A **matrix Lie group** is a subgroup G of $GL(n;\mathbb{C})$ such that if A_m is a sequence of matrices in G, we have $A_m \to A$ implies $A \in G$.

$SL(2;\mathbb{C})$ as a matrix Lie group

Definition

A **matrix Lie group** is a subgroup G of $GL(n;\mathbb{C})$ such that if A_m is a sequence of matrices in G, we have $A_m \to A$ implies $A \in G$.

Example

The group of 2×2 matrices with entries in $\mathbb{C}$ and determinant 1, denoted $SL(2;\mathbb{C})$, is a matrix Lie group.

Proof.

$SL(2;\mathbb{C}) \leq GL(2;\mathbb{C})$ is immediate. Now take a sequence $A_m \to A$ and note that

$$1 = \lim(\det A_m) = \det(\lim A_m) = \det A$$

by continuity of the determinant. □

Lie algebras

Definition

A **Lie algebra** is a finite-dimensional vector space $\mathfrak{g}$ with a map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

1. $[\cdot, \cdot]$ is linear in both arguments separately
2. $[X, Y] = -[Y, X]$ for $X, Y \in \mathfrak{g}$ (skew symmetry)
3. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for $X, Y, Z \in \mathfrak{g}$ (Jacobi identity)

$\mathfrak{sl}(n;\mathbb{C})$ as a Lie algebra

Example

Consider the vector space given by $X \in M_n(\mathbb{C})$ for which $\mathrm{tr}(X) = 0$. Denote this space by $\mathfrak{sl}(n;\mathbb{C})$. Then $\mathfrak{sl}(n;\mathbb{C})$ is a Lie algebra with bracket $[X, Y] = XY - YX$.

Proof.

For $X, Y \in M_n(\mathbb{C})$, we have

$$\mathrm{tr}(XY) = \mathrm{tr}(YX)$$
$$\Rightarrow 0 = \mathrm{tr}(XY) - \mathrm{tr}(YX) = \mathrm{tr}(XY - YX)$$

So $[X, Y] \in \mathfrak{sl}(n;\mathbb{C})$. We can quickly check bilinearity and skew symmetry. Verify the Jacobi identity by direct computation. □

The Lie group-Lie algebra correspondence

Definition

Take a matrix Lie group G. Then the **Lie algebra of** G is the set $\mathfrak{g} := \{X \mid \forall t \in \mathbb{R} : e^{tX} \in G\}$.

The Lie algebra of a Lie group is a linearization of its group action.

$\mathfrak{sl}(n;\mathbb{C})$ and $SL(n;\mathbb{C})$

Claim

$\mathfrak{sl}(n;\mathbb{C})$ is the Lie algebra of $SL(n;\mathbb{C})$!

Proof sketch.

Using the exponential of $X \in M_n(\mathbb{C})$, defined by

$$e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!},$$

we can show $\det(e^X) = e^{\mathrm{tr}(X)}$. So $\mathrm{tr}(X) = 0$ implies $\det(e^{tX}) = 1$ and $X \in \mathfrak{sl}(n;\mathbb{C})$. If $\det(e^{tX}) = e^{t\cdot\mathrm{tr}(X)} = 1$ for all t, then

$$\mathrm{tr}(X) = \frac{d}{dt} e^{t\cdot\mathrm{tr}(X)}\bigg|_{t=0} = 0.$$

□

Homomorphisms

Definition

A **Lie group homomorphism** between matrix Lie groups G, H is a continuous group homomorphism between G and H.

Definition

For Lie algebras $\mathfrak{g}, \mathfrak{h}$, a **Lie algebra homomorphism** is a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ such that $\phi([X, Y]) = [\phi(X), \phi(Y)]$.

Theorem

Given a Lie group homomorphism $\Phi : G \to H$, there exists a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ such that $\Phi(e^X) = e^{\phi(X)}$. It is given by

$$\phi(X) = \frac{d}{dt}\Phi(e^{tX})\Big|_{t=0}.$$

Section 2

Representation theory

Representations

Definition

For a matrix Lie group G, a **finite-dimensional representation** of G is a Lie group homomorphism

$$\Pi : G \to GL(V)$$

for some finite-dimensional vector space V.

Definition

Take a Lie algebra $\mathfrak{g}$. A **finite-dimensional representation** of $\mathfrak{g}$ is a Lie algebra homomorphism

$$\pi : \mathfrak{g} \to \mathfrak{gl}(V)$$

for some finite-dimensional vector space V.

Definition

An **irreducible representation** acting on V is one whose only invariant subspaces are $\{0\}$ and V.

A representation of $\mathfrak{sl}(2;\mathbb{C})$

Consider the space V_m of homogeneous, degree-m polynomials in two complex variables. For $X \in SL(2;\mathbb{C})$, take $\Pi_m(X)$ given by

$$[\Pi_m(X)f](z) = f(X^{-1}z) \qquad z \in \mathbb{C}^2.$$

Check by direct computation that $\Pi_m(X) \in \mathrm{End}(V_m)$.

Claim

Π_m is a representation of $SL(2;\mathbb{C})$.

Proof.

Enough to show Π_m is a homomorphism. Take $X, Y \in SL(2;\mathbb{C})$.

$$\Pi_m(X)[\Pi_m(Y)f](z) = [\Pi_m(Y)f](X^{-1}z) = f(Y^{-1}X^{-1}z)$$
$$= [\Pi_m(XY)f](z).$$

□

A representation of $\mathfrak{sl}(2;\mathbb{C})$ (cont.)

Recall this result:

Theorem

Given a Lie group homomorphism $\Phi : G \to H$, there exists a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ such that $\Phi(e^X) = e^{\phi(X)}$. It is given by

$$\phi(X) = \left.\frac{d}{dt}\Phi(e^{tX})\right|_{t=0}.$$

This gives us a representation of $\mathfrak{sl}(2;\mathbb{C})$!

$$[\pi_m(X)f](z) = \left.\frac{d}{dt}f(e^{-tX}z)\right|_{t=0}.$$

Section 3

Representations of $\mathfrak{sl}(2; \mathbb{C})$

The main result

Theorem

- *For* $m \geq 0$, *there exists an irreducible representation of* $\mathfrak{sl}(2;\mathbb{C})$ *with dimension* $m+1$.
- *Any two irreducible complex representations of* $\mathfrak{sl}(2;\mathbb{C})$ *are isomorphic.*
- *Any irreducible representation of* $\mathfrak{sl}(2;\mathbb{C})$ *with dimension* $m+1$ *is isomorphic to* π_m.

An eigenvector lemma

For the remaining slides, we use the following basis of $\mathfrak{sl}(2;\mathbb{C})$:

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Lemma

Let u be an eigenvector of $\pi(H)$ with eigenvalue α. Then

$$\pi(H)\pi(X)u = (\alpha + 2)\pi(X)u$$

$$\pi(H)\pi(Y)u = (\alpha - 2)\pi(Y)u.$$

So either $\pi(X)u$ is an eigenvector of $\pi(H)$ with eigenvalue $\alpha + 2$, or $\pi(X) = 0$. Analogously for $\pi(Y)u$.

Proof of the lemma

Proof.

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

By linearity of π, we write $[\pi(H), \pi(X)] = \pi([H, X]) = 2\pi(X)$. So

$$\pi(H)\pi(X)u - \pi(X)\pi(H)u = \pi([H, X])u = 2\pi(X)u$$

$$\Rightarrow \pi(H)\pi(X)u = \pi(X)\pi(H)u + 2\pi(X)u$$
$$= (\alpha + 2)\pi(X)u.$$

We argue similarly for $\pi(Y)$. □

Proof of the result

Proof of theorem

Take any irreducible representation π of $\mathfrak{sl}(2;\mathbb{C})$ acting on a finite-dimensional vector space. Let u be an eigenvector of $\pi(H)$ with eigenvalue $\alpha \in \mathbb{C}$. Then

$$\pi(H)\pi(X)^k u = (\alpha + 2k)\pi(X)^k u.$$

Pick $N \geq 0$ such that

$$\pi(X)^N u \neq 0 \qquad \pi(X)^{N+1} u = 0.$$

Define $u_0 = \pi(X)^N u$ and $\lambda = \alpha + 2N$ so that $\pi(H)u_0 = \lambda u_0$ and $\pi(X)u_0 = 0$.

Proof of the result (cont.)

Proof of theorem

Also, define $u_k = \pi(Y)^k u_0$ for $k \geq 0$. Then

$$\pi(H)u_k = (\lambda - 2k)u_k. \quad (1)$$

We can check via induction that

$$\pi(X)u_k = k[\lambda - (k-1)]u_{k-1}. \quad (3)$$

Now pick $m \geq 0$ such that

$$\pi(Y)^k u_0 \neq 0 \ \forall k \leq m \qquad \pi(Y)^{m+1} u_0 = 0. \quad (2)$$

Then

$$0 = \pi(X)u_{m+1} = (m+1)(\lambda - m)u_m.$$

That is, $\lambda = m$.

Proof of the result (cont.)

Proof of theorem

So far, we have

$$\pi(H)u_k = (\lambda - 2k)u_k \qquad (1)$$

$$\pi(Y)^k u_0 \neq 0 \ \forall k \leq m \qquad \pi(Y)^{m+1} u_0 = 0 \qquad (2)$$

$$\pi(X)u_k = k[\lambda - (k-1)]u_{k-1}. \qquad (3)$$

Equivalently, there exists m such that:

$$\pi(H)u_k = (m - 2k)u_k$$

$$\pi(Y)u_k = \begin{cases} u_{k+1} & k < m \\ 0 & k = m \end{cases}$$

$$\pi(X)u_k = \begin{cases} k[m - (k-1)]u_{k-1} & k > 0 \\ 0 & k = 0. \end{cases}$$

Proof of the result (cont.)

Proof of theorem.

$\mathrm{span}(u_0, \ldots, u_m)$ is invariant under $\pi(H), \pi(X), \pi(Y)$, and thus also $\pi(Z)$ for $Z \in \mathfrak{sl}(2; \mathbb{C})$. So $\mathrm{span}(u_0, \ldots, u_m) = V \Rightarrow \dim V = m + 1$.

If we define $\pi(H), \pi(X), \pi(H)$ using the $\{u_k\}$, we can check that the resulting π is a representation of $\mathfrak{sl}(2; \mathbb{C})$.

Hence any irreducible representation with dimension $m + 1$, including π_m, looks like the one above.

This completes the proof! $\square$

A picture of π

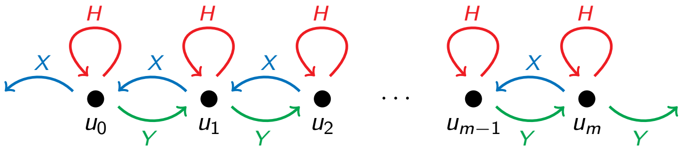
Recall:

$$\pi(H)u_k = (m-2k)u_k$$

$$\pi(Y)u_k = \begin{cases} u_{k+1} & k < m \\ 0 & k = m \end{cases}$$

$$\pi(X)u_k = \begin{cases} k[m-(k-1)]u_{k-1} & k > 0 \\ 0 & k = 0. \end{cases}$$

The basis of $\mathfrak{sl}(2;\mathbb{C})$ acts on the basis of V given by $\{u_k\}$ as such:



Thank you!